

Constraints upon the spectral indices of relic gravitational waves by LIGO S5

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Abstract

With LIGO having achieved its design sensitivity and the LIGO S5 strain data being available, constraints on the relic gravitational waves (RGWs) becomes realistic. The analytical spectrum of RGWs generated during inflation depends sensitively on the initial condition, which is generically described by the index β , the running index α_t , and the tensor-to-scalar ratio r . By the LIGO S5 data of the cross-correlated two detectors, we obtain constraints on the parameters (β, α_t, r) . As a main result, we have computed the theoretical signal-to noise ratio (SNR) of RGWs for various values of (β, α_t, r) , using the cross-correlation for the given pair of LIGO detectors. The constraints by the indirect bound on the energy density of RGWs by BBN and CMB have been obtained, which turn out to be still more stringent than LIGO S5.

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1. Introduction

Recently, LIGO S5 has experimentally obtained so far the most stringent bound on the spectral energy density of the stochastic background of gravitational waves, $\Omega_0 < 6.9 \times 10^{-6}$ around $\sim 100\text{Hz}$ [1]. Generated during inflation, RWGs is of cosmological origin, and has long been investigated [2, 3, 4, 5], and, in particular, its analytical spectrum has been known [6]. It depends most sensitively upon the initial condition, which can be generically summarized by the initial amplitude, the spectral index β , as well as the running index α_t . In particular, small variations of β and α_t will cause substantial change of the amplitude in higher frequencies [7]. The value of β and α_t are usually predicted by specific inflationary models [8] with possible modifications by quantum field renormalization [9]. After inflation, RGWs is altered substantially only by a sequence of subsequent expansions, the reheating, the radiation era, the matter era, and the current acceleration era [6], essentially unaffected by the cosmic matter they encounter. As a result, RGWs carry a unique information of the early Universe, and can probe the Universe much earlier than the cosmic microwave background (CMB). Such cosmic processes, as neutrino free-streaming [10, 11], QCD

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transition, and e^+e^- annihilation [12], etc, affect RGWs less substantially than small variations of β and α_t around the frequency range $\sim 100\text{Hz}$ of LIGO, and can be neglected in this study.

Spreading over a broad range of frequency, ($10^{-18} \sim 10^{10}$) Hz, RGWs is a major target of detectors working at various frequencies, including LIGO [13], Advanced LIGO [14], LISA [15], EXPLORER [16], millisecond pulsar timing [17], and Gauss Beam [18], etc. The curl type of CMB polarization is only contributed by RGWs, measurements of which also serve as detectors [19], such as WMAP [20, 21, 22, 23, 24], Planck [25] and CMBpol [26]. Prior to the LIGO S5 bound [1], often used were the bound from big bang nucleosynthesis (BBN) [27] and that from the CMB anisotropy spectrum [28]. These two indirect bounds actually constrain the energy density $\int \Omega_g(f) d \ln f$ as an integration of the spectral energy density $\Omega_g(f)$. By contrast, the LIGO S5 bound is upon $\Omega_g(f)$ itself, and has surpassed the LIGO S4 [29] by more than an order of magnitude. It is now realistic to infer from this bound some constraints on the initial condition of RGWs in terms of (β, α_t, r) . In this letter, using the strain data from LIGO S5 [1], we will derive such constraints, and compute the theoretical SNR for the analytic spectrum of RGWs with various (β, α_t, r) .

2. Analytical Spectrum of RGWs

In a spatially flat Robertson-Walker spacetime, the analytical mode $h_k(\tau)$ of RGWs is known [6]. The spectrum at the present time τ_H is given by

$$h(k, \tau_H) \equiv \frac{\sqrt{2}}{\pi} k^{3/2} |h_k(\tau_H)|, \quad (1)$$

related to the characteristic amplitude [30], $h_c(f) = h(k, \tau_H)/\sqrt{2}$. Here the frequency f is related to the wavenumber k via $f = kH_0/2\pi\gamma$ with $\gamma \simeq 1.97$ for $\Omega_\Lambda \simeq 0.73$ [11]. The spectral energy density [3, 7]

$$\Omega_g(f) = \frac{2\pi^2}{3} h_c^2(f) \left(\frac{f}{H_0} \right)^2, \quad (2)$$

where $H_0 = 3.24 h \times 10^{-18}$ Hz. RGWs is completely fixed, once the initial condition is given, which is taken at the time τ_i of the horizon-crossing during the inflation, of a generic form [7, 20, 22]

$$h(k, \tau_i) = \Delta_{\mathcal{R}}(k_0) r^{\frac{1}{2}} \left(\frac{k}{k_0} \right)^{2+\beta+\frac{1}{4}\alpha_t \ln(k/k_0)}, \quad (3)$$

where the pivot wavenumber k_0 corresponds to a physical wavenumber 0.002 Mpc^{-1} , the tensor-to-scalar ratio $r \equiv \Delta_h^2(k_0)/\Delta_{\mathcal{R}}^2(k_0)$ is a re-parametrization of the normalization $\Delta_h(k_0)$ with $\Delta_{\mathcal{R}}^2(k_0) = (2.445 \pm 0.096) \times 10^{-9}$ by WMAP5+BAO+SN Mean [22], the index β is related to the index of the power-law scale factor during inflation $a(\tau) \propto |\tau|^{1+\beta}$ [3, 6] and $\beta \simeq -2$ yields a nearly scale-invariant spectrum, and the running index α_t reflects an extra bending. Observations of CMB anisotropies have given preliminary results on the scalar index and the scalar running index [20, 21, 22]. So far there is no observation of β and α_t , and there are only some upper bound on r [22, 23, 24]. In scalar inflationary models, β and α_t are determined by the inflation potential and its derivatives [8]. There might be relations between the tensorial indices and the scalar ones. For generality, we treat (β, α_t, r) as independent parameters. In literature the notation n_t is often used, $n_t = 2\beta + 4$.

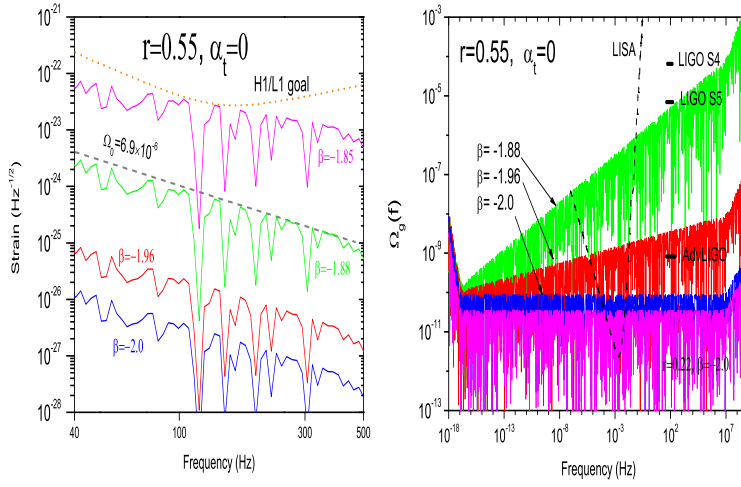


Figure 1: Left: The spectrum $h_c(f)\sqrt{F}/\sqrt{2f}$ with $r = 0.55$ and $\alpha_t = 0$ for various β . The dot line (labeled by H1/L1 goal) is the strain sensitivity \tilde{h}_f of single interferometers achieved by LIGO S5, and the dash line ($\Omega_0 = 6.9 \times 10^{-6}$) is the corresponding sensitivity from cross-correlated two interferometers of LIGO S5 [1]. Right: The spectral energy density $\Omega_g(f)$ with $\alpha_t = 0$ for various β . The upper limit of LIGO S5 corresponds to the dash line in the left.

3. Constraints on the spectral indices of RGWs

The left panel of Fig. 1 gives the analytic spectrum $h_c(f)\sqrt{F}/\sqrt{2f}$ of RGWs in the frequency range (40, 500) Hz for various β in the model $r = 0.55$ and $\alpha_t = 0$. The irregular oscillations in the curves of the analytic spectra are due to the combinations of Bessel functions implicitly contained in the analytic solution of RGWs [6, 7]. It is seen that a small variation in β from -2.0 to -1.85 leads to an enhancement of amplitude of $h_c(f)$ by 4 orders of magnitude around ~ 100 Hz. For RGWs to be detectable by a single detector with a strain sensitivity \tilde{h}_f , the condition is [30],

$$\frac{h_c(f)}{\sqrt{2f}}\sqrt{F} \geq \tilde{h}_f, \quad (4)$$

where the angular factor $F = 2/5$ for one interferometer. The dot line (labeled by H1/L1 goal) in the upper part of the left of Fig.1 is the single-detector strain sensitivity achieved by H1 and L1 of LIGO S5 [1]. Thus, we have plotted $h_c(f)\sqrt{F}/\sqrt{2f}$ to directly compare with the strain \tilde{h}_f . The single interferometers, H1 and L1, of the LIGO S5 put a constraint on the index: $\beta \leq -1.85$ for the model $r = 0.55$ and $\alpha_t = 0$.

However, by the cross-correlation of two interferometers, H1 and L1, of the LIGO S5, the detectability is much improved. Approximately, in a narrow band Δf of frequencies and a duration T of observation, the detectability condition is schematically changed to [30]

$$\frac{h_c(f)}{\sqrt{2f}}\sqrt{F} > \frac{1}{(2T\Delta f)^{1/4}}\tilde{h}_f, \quad (5)$$

where \tilde{h}_f is the strain of single detector. For T being long enough so that $(2T\Delta f)^{1/4} \gg 1$, the right hand side of Eq.(5) will be reduced considerably. A detailed description of quantitative treatment is given in Ref.[31]. For the case of a flat spectral energy density Ω_0 , the effective strain of LIGO S5, plotted in the dash line in left of Fig.1, is ~ 100 times lower than that from the single interferometers [1]. This upper limit leads to a more stringent constraint on the index: $\beta \leq -1.88$ for the same model. This is consistent

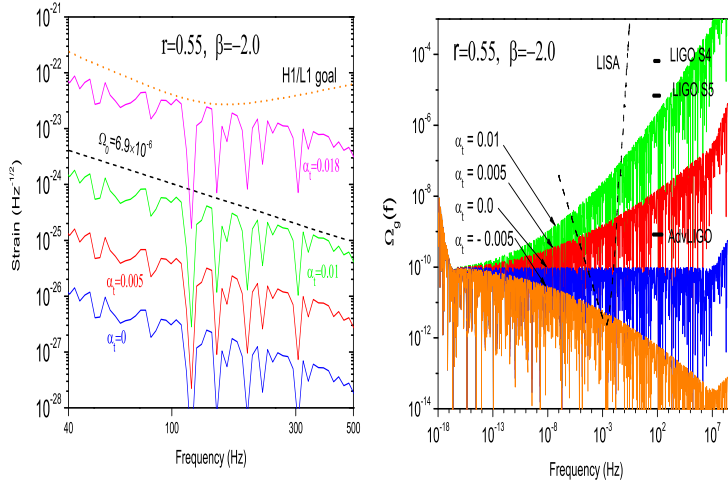


Figure 2: Left: $h_c(f)\sqrt{F}/\sqrt{2f}$ with $r = 0.55$ and $\beta = -2.0$ for various α_t . Right: $\Omega_g(f)$ with $r = 0.55$ and $\beta = -2.0$ for various α_t .

with the current observational result of the scalar index n_s ranging over $(0.97 \sim 1.2)$ [20, 21, 22], if a relation $n_s = 2\beta + 5$ is adopted, as in scalar inflationary models.

The right of Fig. 1 gives the spectral energy density $\Omega_g(f)$ that corresponds to the respective spectrum $h_c(f)$ in the left. By the upper limit $\Omega_0 = 6.9 \times 10^{-6}$ from cross-correlated interferometers of LIGO S5, the resulting constraint is $\beta \leq -1.88$, the same as from the left. Except for the model $\beta = -2.0$ and $\alpha_t = 0$, $\Omega_g(f)$ is generally not flat, and a larger β leads to a higher amplitude of $\Omega_g(f)$ in higher frequencies [3, 7]. $\Omega_g(f)$ behaves approximately as $\Omega_g(f) \propto f^{0.24}$ for the model $\beta = -1.88$ and $\alpha_t = 0$. For comparison, the sensitivity of LISA is plotted and has a broader frequency range.

The left of Fig. 2 plots $h_c(f)\sqrt{F}/\sqrt{2f}$ for various α_t in the model of $r = 0.55$ and $\beta = -2.0$. A small variation in α_t from 0 to 0.018 enhances the amplitude of $h_c(f)$ by ~ 4 orders of magnitude around ~ 100 Hz. The single interferometers of the LIGO S5 puts a constraint on the running index: $\alpha_t \leq 0.018$. The cross-correlation of two interferometers of the LIGO S5 puts a more stringent constraint: $\alpha_t \leq 0.01$. So far the preliminary observed result of the scalar running index α_s ranges over $(-0.050 \sim -0.077)$ by WMAP [20, 21, 22]. If both RGWs and scalar perturbations are generated by the same inflation, one expects α_t to be nearly as small as α_s for several kinds of smooth scalar potential [8]. If so, the constraint on α_t by LIGO S5 is consistent with the results by WMAP. The right of Fig. 2 gives $\Omega_g(f)$ that corresponds to those in the left. The upper limit of LIGO S5 gives the constraint $\alpha_t \leq 0.01$, same as that from the left. For the model $\beta = -2.0$ and $\alpha_t = 0.01$, the slope is $\Omega_g(f) \propto f^{0.45}$, not flat either.

Figure 3 shows that, around ~ 100 Hz, the model $\beta = -2.0$ and $\alpha_t = 0.011$ and the model $\beta = -1.88$ and $\alpha_t = 0$ yield the same height of amplitude detectable by LIGO S5. Moreover, the slopes of $\Omega_g(f)$ in the two models only differ slightly. Therefore, there is a degeneracy between the indices β and α_t . Given a rather narrow frequency range, $(41.5, 169.25)$ Hz, it is unlikely for LIGO S5 to distinguish the spectra from these two models. Comparatively, LISA with a much broader frequency range would have consequently a better chance to distinguish models with different β and α_t .

The above examinations on detectability via comparison of the spectrum $h_c(f)$ and the strain \tilde{h}_f are

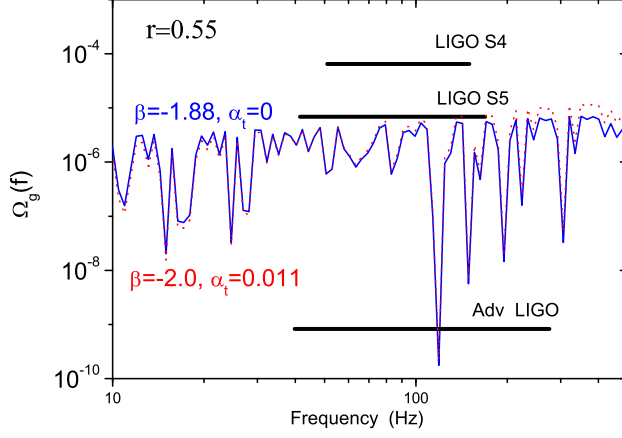


Figure 3: $\Omega_g(f)$ has the same height at 100Hz for the models with $\beta = -2.0$ and $\alpha_t = 0.011$, and with $\beta = -1.88$ and $\alpha_t = 0$, respectively.

Table 1: The SNR for RGWs with $r=0.1$ for the given pair of detectors of LIGO S5

	$\alpha_t = 0$	$\alpha_t = 0.005$	$\alpha_t = 0.007$	$\alpha_t = 0.01$
$\beta = -2.0$	5.4×10^{-6}	8.0×10^{-4}	6.0×10^{-3}	1.2×10^{-1}
$\beta = -1.96$	2.0×10^{-4}	3.0×10^{-2}	2.2×10^{-1}	4.5
$\beta = -1.90$	4.5×10^{-2}	6.7	5.0×10^1	1.0×10^3
$\beta = -1.88$	2.8×10^{-1}	4.1×10^1	3.0×10^2	6.2×10^3

still qualitative. According to the method developed in Ref.[31], a more quantitative description of the detectability is through the signal-noise ratio

$$\text{SNR} = \frac{3H_0^2}{10\pi^2} \sqrt{T} \left[\int_{-\infty}^{\infty} df \frac{\gamma^2(f) \Omega_g^2(f)}{f^6 P_1(f) P_2(f)} \right]^{1/2} \quad (6)$$

for the given pair of detectors of LIGO, where $P_1(f)$ and $P_2(f)$ are the noise power spectrum of detector, H1 and L1, respectively [1], and $\gamma(f)$ is the overlap reduction function [31]. Since the data of the strain sensitivity $\tilde{h}_{f1} = \sqrt{P_1(f)}$ and $\tilde{h}_{f2} = \sqrt{P_2(f)}$ have been given [1], it is straightforward to calculate SNR from $\Omega_g(f)$ for each model. For the model $\Omega_\Lambda = 0.73$ and $\Omega_m = 0.27$, we have computed the corresponding SNR for various indices β , and α_t , listed in Table 1 for $r = 0.1$. The duration T in Eq.(6) for LIGO S5 is from Nov. 5, 2005 to Sep. 30, 2007 [1], i.e., $T = 59961600$ seconds. Clearly, greater values of β and α_t yield higher SNR accordingly. For other values of r , the corresponding SNR follows immediately since $\text{SNR} \propto r$.

3. Constraints via the energy density Ω_{gw}

Before LIGO S5 data is available in constraining the spectrum $\Omega_g(f)$, often used is the energy density parameter

$$\Omega_{gw} = \int_{f_{low}}^{f_{upper}} \Omega_g(f) \frac{df}{f}, \quad (7)$$

as an integration of $\Omega_g(f)$ over certain frequency range, where the cutoffs of frequencies depend on specific situation under consideration. For the total energy density of RGWs in the universe, one can take $f_{low} \simeq 2 \times 10^{-18}$ Hz and $f_{upper} \simeq 10^{10}$ Hz [11]. Strictly speaking, limits coming out of this method do not apply to the spectrum $\Omega_g(f)$, and are of indirect nature. Sometimes $\Omega_g(f)$ and Ω_{gw} were used indiscriminatingly in literature. But this will be valid only under the condition that the integration interval $df/f = d\ln f \sim 1$ and that $\Omega_g(f)$ be nearly frequency-independent (flat), which is not the case for general indices β and α_t , as has been demonstrated earlier. Whenever possible, one should distinguish $\Omega_g(f)$ and Ω_{gw} for a pertinent treatment. Currently, two observed bounds on Ω_{gw} are available. One is $\Omega_{gw} < \Omega_{BBN} \equiv 1.1 \times 10^{-5}(N_\nu - 3)$ from BBN, where N_ν is the effective number of relativistic species at the time of BBN. The abundances of light-element, combined with WMAP data, give $(N_\nu - 3) < 1.4$ [27], so $\Omega_{BBN} = 1.5 \times 10^{-5}$ [29]. This bound receives contribution from frequencies down to the lower limit $f_{low} \sim 10^{-10}$ Hz, corresponding to the horizon scale at the time of BBN [31]. Another bound is $\Omega_{gw} < \Omega_{CMB}h^2 \equiv 8.4 \times 10^{-6}$ at 95% C.L. from CMB + matter power spectrum + Ly α for the homogeneous initial condition of RGWs [28]. For the Hubble parameter $h = 0.701$ [22, 23], this is $\Omega_{CMB} = 1.62 \times 10^{-5}$, receiving contributions from frequencies down to a much lower limit $f_{low} \sim 10^{-15}$ Hz, corresponding to the horizon scale at the decoupling for CMB. From the theoretical side, substituting the analytical spectrum $\Omega_g(f)$ as the integrand into Eq.(7), the resulting integral Ω_{gw} is a function of the indices β and α_t , since $\Omega_g(f)$ intrinsically depends on β and α_t . By this way, we can derive constraints on β and α_t by the bounds Ω_{BBN} and Ω_{CMB} . In carrying out the integration, we take the upper limit of integration $f_{upper} = 10^{10}$ Hz [11]. As for the lower limit, we take $f_{low} = 10^{-10}$ Hz for BBN case, and $f_{low} = 10^{-15}$ Hz for CMB case, respectively. It turns out that the integral Ω_{gw} is sensitive to the value of f_{low} for very small β and α_t .

The left of Fig.4 shows the β -dependence of Ω_{gw} for fixed $\alpha_t = 0$ and $r = 0.55$ and 0.1 , and the right shows the α_t -dependence of Ω_{gw} for fixed $\beta = -2.0$ and $r = 0.55$ and 0.1 . In Fig.4 the horizontal dash lines are the bounds Ω_{BBN} and Ω_{CMB} , which are close to each other. The resulting constraint on β is $\beta \lesssim -1.96$ for $r = 0.55$ and $\alpha_t = 0$, and $\beta \lesssim -1.98$ for $r = 0.1$ and $\alpha_t = 0$. The resulting constraint on α_t is $\alpha_t \lesssim 0.004$ for $r = 0.55$ and $\beta = -2.0$, and $\alpha_t \lesssim 0.005$ for $r = 0.1$ and $\beta = -2.0$. These constraints on β and α_t by BBN and CMB are more stringent than those by LIGO S5.

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References

- [1] The LIGO Collaboration and The VIRGO Collaboration, *Nature* **460**, 990 (2009).
- [2] L. P. Grishchuk, *Sov.Phys.JETP* **40**, 409 (1975);
- [3] L. P. Grishchuk, in *Lecture Notes in Physics*, Vol.562, p.167, Springer-Verlag, (2001), arXiv: gr-qc/0002035; arXiv: gr-qc/0707.3319.
- [4] A. A. Starobinsky, *JEPT Lett.* **30**, 682 (1979);

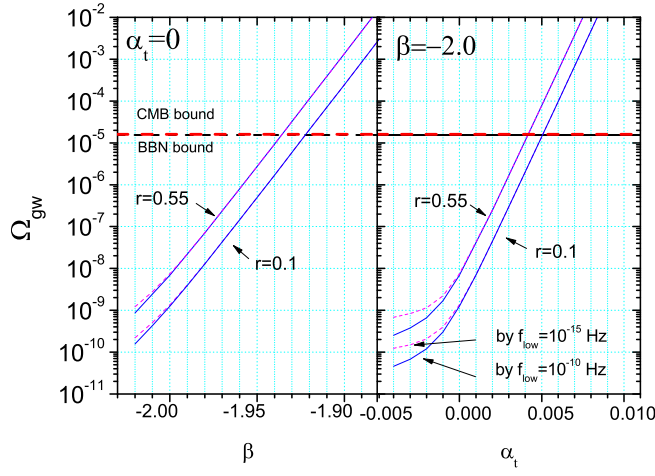


Figure 4: Left: Ω_{gw} as a function of β . Right: Ω_{gw} as a function of α_t .

- [5] V. A. Rubakov, M.Sazhin, and A. Veryaskin, Phys.Lett.B **115**, 189 (1982);
L. F. Abbott & M.B. Wise, Nucl.Phys.B**244**, 541 (1984);
A. A. Starobinsky, Sov.Astron.Lett.**11**, 133 (1985);
B. Allen, Phys.Rev.D**37**, 2078 (1988);
V. Sahni, Phys.Rev.D**42**, 453 (1990);
W.Zhao and Y.Zhang, Phys.Rev.D**74**, 043503 (2006).
- [6] Y. Zhang *et al.*, Class. Quant. Grav. **22**, 1383 (2005); Chin. Phys. Lett. **22**, 1817 (2005); Class. Quant. Grav.**23**, 3783 (2006).
- [7] M.L. Tong, Y. Zhang, Phys.Rev.D**80**, 084022 (2009).
- [8] A. Liddle and D. Lyth, Phys.Lett. B**291**, (1992) 391;
A.R. Liddle and M.S. Turner, Phys. Rev. D**50**, 758 (1994);
A. Kosowsky and M.S. Turner, Phys. Rev. D**52**, R1739 (1995).
- [9] I. Agullo, J. Navarro-Salas, G.J.Olmo, L.Parker, Phys. Rev. Lett.**101**, 171301 (2008); Phys. Rev. Lett.**103**, 061301 (2009).
- [10] S. Weinberg, Phys. Rev. D**69**, 023503 (2004);
Y. Watanabe and E. Komatsu, Phys. Rev. D**73**, 123515 (2006).
- [11] H. X. Miao and Y. Zhang, Phys. Rev. D **75**, 104009 (2007).
- [12] S. Wang, Y. Zhang, T.Y. Xia, and H.X. Miao, Phys. Rev. D **77**, 104016 (2008).
- [13] <http://www.ligo.caltech.edu/>
- [14] <http://www.ligo.caltech.edu/advLIGO>

- [15] <http://lisa.nasa.gov/>
<http://www.srl.caltech.edu/~shane/sensitivity/MakeCurve.html>
- [16] P. Astone, *et al.*, Class. Quant. Grav.**25**, 114028 (2008).
- [17] S.E. Thorsett and R.J. Dewey, Phys. Rev. D**53**, 3468 (1996).
 G. Hobbs, Class. Quant. Grav.**25**, 114032 (2008);
- [18] M.L. Tong, Y. Zhang, and F.Y. Li, Phys. Rev. D **78**, 024041 (2008).
- [19] M. Zaldarriaga and U. Seljak, Phys.Rev.D**55**, 1830 (1997);
 M. Kamionkowski, A. Kosowsky, and A. Stebbins, Phys. Rev. D**55**, 7368 (1997);
 B.G. Keating, P.T. Timbie, A. Polnarev, and J. Steinberger, Astrophys. J. **495**, 580 (1998);
 J. R. Pritchard and M. Kamionkowski, Ann. Phys.(N.Y.) **318**, 2 (2005);
 W. Zhao and Y. Zhang, Phys.Rev.D**74**, 083006 (2006);
 T.Y Xia and Y. Zhang, Phys. Rev. D**78**, 123005 (2008); Phys. Rev. D**79**, 083002 (2009).
- [20] H.V. Peiris, *et al.*, Astrophys. J. Suppl. **148**, 213 (2003).
 D.N. Spergel, *et al.*, Astrophys. J. Suppl. **148**, 175 (2003).
- [21] D.N. Spergel, *et al.*, Astrophys. J. Suppl. **170**, 377 (2007).
 L. Page, *et al.*, Astrophys.J.Suppl. **170**, 335 (2007).
- [22] E. Komatsu, *et al.*, Astrophys. J. Suppl. **180**, 330 (2009).
- [23] G. Hinshaw, *et al.*, Astrophys. J. Suppl. **180**, 225 (2009);
- [24] J. Dunkley, *et al.*, Astrophys. J. Suppl. **180**, 306 (2009).
- [25] <http://www.rssd.esa.int/index.php?project=Planck>
- [26] D. Baumann *et al.*, arXiv:0811.3919;
 M. Zaldarriaga *et al.*, arXiv:0811.3918
- [27] R.H. Cyburt, B.D. Fields, K.A. Olive, and E. Skillman, Astropart. Phys.**23**, 313 (2005).
- [28] T.L. Smith, E. Pierpaoli, and M. Kamionkowski, Phys. Rev. Lett. **97**, 021301 (2006).
- [29] B. Abbott, *et al.*, Astrophys. J. **659**, 918 (2006).
- [30] M. Maggiore, Phys. Rept.**331**, 283 (2000).
- [31] B. Allen, arXiv: gr-qc/9604033 (1996); B. Allen and J.D. Romano, Phys. Rev. D **59**, 102001 (1999).